

Non-Commutative Gebauer-Möller Criteria

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Abstract

For an efficient implementation of Buchberger's Algorithm, it is essential to avoid the treatment of as many unnecessary critical pairs or obstructions as possible. In the case of the commutative polynomial ring, this is achieved by the Gebauer-Möller criteria. Here we present an adaptation of the Gebauer-Möller criteria for non-commutative polynomial rings, i.e. for free associative algebras over fields. The essential idea is to detect unnecessary obstructions using other obstructions with or without overlap. Experiments show that the new criteria are able to detect almost all unnecessary obstructions during the execution of Buchberger's procedure.

Keywords: Gröbner basis, free associative algebra, obstruction, Buchberger procedure

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1 Introduction

Ever since B. Buchberger's thesis [2], Gröbner bases have become a fundamental tool for computations in commutative algebra and algebraic geometry. The most time-consuming part in Buchberger's Algorithm is the computation of the normal remainder of an S-polynomial corresponding to a critical pair. Therefore a significant amount of energy has been spent on reducing the number of critical pairs which have to be treated. After the discovery of various criteria for discarding critical pairs ahead of time by B. Buchberger and H.M. Möller (see [3], [4] and [10]), this subject found an initial resolution via the *Gebauer-Möller installation* presented in [7] which offers a good compromise between efficiency and the success rate for detecting unnecessary critical pairs.

A very different picture presents itself for Gröbner basis computations for two-sided ideals in non-commutative polynomial rings. The basic Gröbner basis theory in this case was described by T. Mora and others (see [11] and [12]), and obstructions, the non-commutative analogue of critical pairs, were studied in [12]. However, since very few authors endeavoured to implement efficient versions of Buchberger's Procedure for the non-commutative polynomial ring (i.e.

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the free associative algebra), the subject of minimizing the number of obstructions which have to be treated has received comparatively little attention, and merely a few rules were developed. (For an overview, see for instance [6]).

In this paper, we present generalizations of the Gebauer-Möller criteria for non-commutative polynomials. They cover not only the known cases of useless obstructions discussed in [12], Lemma 5.5 and [6], but form a complete analogue of the results in the commutative case. One of the key ingredients we use for this purpose is the consideration of obstructions without overlaps. We detect useless obstructions, i.e. obstructions that can be represented by other obstructions, via not only obstructions with overlaps but also those without overlaps. We show that the consideration of obstructions without overlaps won't increase unnecessary computations, since a Gröbner representation is inherent in the S-polynomial of every obstruction without overlaps. Consequently, we reduce the number of obstructions efficiently and obtain a non-commutative version of the Gebauer-Möller criteria.

This paper is organised as follows. In Section 2 we recall the basic theory of Gröbner bases for two-sided ideals in non-commutative polynomial rings. In particular, we introduce and study obstructions (see Definitions 2.4 and 2.9, and Lemmas 2.6 and 2.8), present the Buchberger Criterion (see Proposition 2.10), and formulate the Buchberger Procedure (see Theorem 2.11). The non-commutative analogues of the Gebauer-Möller criteria are developed in Section 3. They are based on a careful study of the set of newly constructed obstructions which are produced during the execution of Buchberger's Procedure. As a result, we are able to formulate the Non-Commutative Multiply Criterion (see Proposition 3.7), the Non-Commutative Leading Word Criterion (see Proposition 3.8) and the Non-Commutative Backward Criterion (see Proposition 3.13).

The second author has implemented a version of the Buchberger Procedure for non-commutative polynomial rings in the computer algebra system ApCoCoA which includes the non-commutative Gebauer-Möller criteria developed here (see [1]). In the last section, we present experimental results about the efficiency of the criteria for some cases of moderately difficult Gröbner basis computations.

Unless mentioned otherwise, we adhere to the definitions and terminology given in [8] and [9].

2 Gröbner Bases in $K\langle X \rangle$

In the following we let $X = \{x_1, \dots, x_n\}$ be a finite set of indeterminates (or a finite alphabet), and $\langle X \rangle$ the monoid of all *words* (or *terms*) $x_{i_1} \cdots x_{i_l}$ where the multiplication is concatenation of words. The empty word will be denoted by λ . Furthermore, let K be a field, and let

$$K\langle X \rangle = \{c_1 w_1 + \cdots + c_s w_s \mid s \in \mathbb{N}, c_i \in K \setminus \{0\}, w_i \in \langle X \rangle\}$$

be the non-commutative polynomial ring generated by X over K (or the free associative K -algebra generated by X). We introduce basic notions of Gröbner basis theory in this setting.

Definition 2.1. A *word ordering* on $\langle X \rangle$ is a well-ordering σ which is compatible with multiplication, i.e. $w_1 \geq_\sigma w_2$ implies $w_3 w_1 w_4 \geq_\sigma w_3 w_2 w_4$ for all words $w_1, w_2, w_3, w_4 \in \langle X \rangle$.

In the commutative case, a word ordering is usually called a *term ordering* or *monomial ordering*. For instance, the *length-lexicographic ordering* lllex is a word ordering. It first compares the length of two words and then breaks ties using the non-commutative lexicographic ordering with respect to $x_1 >_{\text{lllex}} \dots >_{\text{lllex}} x_n$. Note that the non-commutative lexicographic ordering by itself is not a word ordering, since it is neither a well-ordering nor compatible with multiplication.

Definition 2.2. Let σ be a word ordering on $\langle X \rangle$.

- (a) Given a polynomial $f \in K\langle X \rangle \setminus \{0\}$, there exists a unique representation $f = c_1 w_1 + \dots + c_s w_s$ with $c_1, \dots, c_s \in K \setminus \{0\}$ and $w_1, \dots, w_s \in \langle X \rangle$ such that $w_1 >_\sigma \dots >_\sigma w_s$. The word $\text{Lw}_\sigma(f) = w_1$ is called the *leading word* of f with respect to σ . The element $\text{Lc}_\sigma(f) = c_1$ is called the *leading coefficient*. We let $\text{Lm}_\sigma(f) = c_1 w_1$ and call it the *leading monomial* of f .
- (b) Let $I \subseteq K\langle X \rangle$ be a two-sided ideal. The set $\text{Lw}_\sigma\{I\} = \{\text{Lw}_\sigma(f) \mid f \in I \setminus \{0\}\} \subseteq \langle X \rangle$ is called the *leading word set* of I . The two-sided ideal $\text{Lw}_\sigma(I) = \langle \text{Lw}_\sigma(f) \mid f \in I \setminus \{0\} \rangle \subseteq K\langle X \rangle$ is called the *leading word ideal* of I .
- (c) A subset G of a two-sided ideal $I \subseteq K\langle X \rangle$ is called a σ -*Gröbner basis* of I if the set of the leading words $\text{Lw}_\sigma\{G\} = \{\text{Lw}_\sigma(f) \mid f \in G \setminus \{0\}\}$ generates the leading word ideal $\text{Lw}_\sigma(I)$.

In the following we focus on computations of Gröbner bases for two-sided ideals in $K\langle X \rangle$. For readers who want to know further properties and applications of non-commutative Gröbner bases, we refer to [12] and [15]. Throughout this paper we assume that σ is a word ordering on $\langle X \rangle$. The next algorithm is a central part of all Gröbner basis computations.

Theorem 2.3. (The Division Algorithm)

Let $f \in K\langle X \rangle$, $s \geq 1$, and $G = \{g_1, \dots, g_s\} \subseteq K\langle X \rangle \setminus \{0\}$. Consider the following sequence of instructions.

- (D1) Let $k_1 = \dots = k_s = 0, p = 0$, and $v = f$.
- (D2) Find the smallest index $i \in \{1, \dots, s\}$ such that $\text{Lw}_\sigma(v) = w \text{Lw}_\sigma(g_i) w'$ for some words $w, w' \in \langle X \rangle$. If such an i exists, increase k_i by 1, set $c_{ik_i} = \frac{\text{Lc}_\sigma(v)}{\text{Lc}_\sigma(g_i)}, w_{ik_i} = w, w'_{ik_i} = w'$, and replace v by $v - c_{ik_i} w_{ik_i} g_i w'_{ik_i}$.

(D3) Repeat step (D2) until there is no more $i \in \{1, \dots, s\}$ such that $\text{Lw}_\sigma(v)$ is a multiple of $\text{Lw}_\sigma(g_i)$. If now $v \neq 0$, then replace p by $p + \text{Lm}_\sigma(v)$ and v by $v - \text{Lm}_\sigma(v)$, continue with step (D2).

(D4) Return the tuples $(c_{11}, w_{11}, w'_{11}), \dots, (c_{sk_s}, w_{sk_s}, w'_{sk_s})$ and the polynomial p .

This is an algorithm which returns tuples $(c_{11}, w_{11}, w'_{11}), \dots, (c_{sk_s}, w_{sk_s}, w'_{sk_s})$ and a polynomial $p \in K\langle X \rangle$ such that the following conditions are satisfied.

- (a) We have $f = \sum_{i=1}^s \sum_{j=1}^{k_i} c_{ij} w_{ij} g_i w'_{ij} + p$.
- (b) No element of $\text{Supp}(p)$ is contained in $\langle \text{Lw}_\sigma(g_1), \dots, \text{Lw}_\sigma(g_s) \rangle$.
- (c) For all $i \in \{1, \dots, s\}$ and all $j \in \{1, \dots, k_i\}$, we have $\text{Lw}_\sigma(w_{ij} g_i w'_{ij}) \leq_\sigma \text{Lw}_\sigma(f)$. If $p \neq 0$, we have $\text{Lw}_\sigma(p) \leq_\sigma \text{Lw}_\sigma(f)$.
- (d) For all $i \in \{1, \dots, s\}$ and all $j \in \{1, \dots, k_i\}$, we have $\text{Lw}_\sigma(w_{ij} g_i w'_{ij}) \notin \langle \text{Lw}_\sigma(g_1), \dots, \text{Lw}_\sigma(g_{i-1}) \rangle$.

Note that the resulting tuples $(c_{11}, w_{11}, w'_{11}), \dots, (c_{sk_s}, w_{sk_s}, w'_{sk_s})$ and polynomial p satisfying conditions (a)-(d) are *not* unique. This is due to the fact that in step (D2) of the Division Algorithm there might exist more than one pair (w, w') satisfying $\text{Lw}_\sigma(v) = w \text{Lw}_\sigma(g_i) w'$ (see [15], Example 3.2.2). A polynomial $p \in K\langle X \rangle$ obtained in Theorem 2.3 is called a *normal remainder* of f with respect to G and is denoted by $\text{NR}_{\sigma, G}(f)$.

For $s \geq 1$, we let $F_s = (K\langle X \rangle \otimes_K K\langle X \rangle)^s$ be the free two-sided $K\langle X \rangle$ -module of rank s with the canonical basis $\{e_1, \dots, e_s\}$, where $e_i = (0, \dots, 0, 1 \otimes 1, 0, \dots, 0)$ with $1 \otimes 1$ occurring in the i^{th} position for $i = 1, \dots, s$, and we let $\mathbb{T}(F_s)$ be the set of terms in F_s , i.e. $\mathbb{T}(F_s) = \{w \epsilon_i w' \mid i \in \{1, \dots, s\}, w, w' \in \langle X \rangle\}$.

Definition 2.4. Let $G = \{g_1, \dots, g_s\} \subseteq K\langle X \rangle \setminus \{0\}$ with $s \geq 1$, and let $i, j \in \{1, \dots, s\}$ such that $i \leq j$.

- (a) If there exist some words $w_i, w'_i, w_j, w'_j \in \langle X \rangle$ such that $w_i \text{Lw}_\sigma(g_i) w'_i = w_j \text{Lw}_\sigma(g_j) w'_j$, then we call the element

$$\text{o}_{i,j}(w_i, w'_i; w_j, w'_j) = \frac{1}{\text{Lc}_\sigma(g_i)} w_i e_i w'_i - \frac{1}{\text{Lc}_\sigma(g_j)} w_j e_j w'_j \in F_s \setminus \{0\}$$

an *obstruction* of g_i and g_j . If $i = j$, it is called a *self obstruction* of g_i . We will denote the *set of all obstructions* of g_i and g_j by $\text{Obs}(i, j)$.

- (b) Let $\text{o}_{i,j}(w_i, w'_i; w_j, w'_j) \in \text{Obs}(i, j)$ be an obstruction of g_i and g_j . The polynomial

$$S_{i,j}(w_i, w'_i; w_j, w'_j) = \frac{1}{\text{Lc}_\sigma(g_i)} w_i g_i w'_i - \frac{1}{\text{Lc}_\sigma(g_j)} w_j g_j w'_j \in K\langle X \rangle$$

is called the *S-polynomial* of $\text{o}_{i,j}(w_i, w'_i; w_j, w'_j)$.

Using these definitions, we can characterize Gröbner bases in the following way.

Proposition 2.5. *Let $G = \{g_1, \dots, g_s\} \subseteq K\langle X \rangle \setminus \{0\}$ be a set of polynomials which generate a two-sided ideal $I = \langle G \rangle \subseteq K\langle X \rangle$. Then the following conditions are equivalent.*

- (a) *The set G is a σ -Gröbner basis of I .*
- (b) *For every obstruction $\text{o}_{i,j}(w_i, w'_i; w_j, w'_j)$ in the set $\bigcup_{1 \leq i \leq j \leq s} \text{Obs}(i, j)$, its S -polynomial $S_{i,j}(w_i, w'_i; w_j, w'_j)$ has a representation*

$$S_{i,j}(w_i, w'_i; w_j, w'_j) = \sum_{k=1}^{\mu} c_k w_k g_{i_k} w'_k$$

with $c_k \in K, w_k, w'_k \in \langle X \rangle$, and $g_{i_k} \in G$ for all $k \in \{1, \dots, \mu\}$ such that $\text{Lw}_{\sigma}(w_j g_j w'_j) >_{\sigma} \text{Lw}_{\sigma}(w_k g_{i_k} w'_k)$ if $c_k \neq 0$ for some $k \in \{1, \dots, \mu\}$.

Proof. See [15], Proposition 4.1.2. □

A presentation of $S_{i,j}(w_i, w'_i; w_j, w'_j)$ as in Proposition 2.5.b is called a (weak) Gröbner representation of $S_{i,j}(w_i, w'_i; w_j, w'_j)$ in terms of G .

Observe that there are infinitely many obstructions in each set $\text{Obs}(i, j)$, due to the following two types of *trivial* obstructions.

- (T1) If $\text{o}_{i,j}(w_i, w'_i; w_j, w'_j) \in \text{Obs}(i, j)$, then, for all $w, w' \in \langle X \rangle$, we have $\text{o}_{i,j}(w w_i, w'_i w'; w w_j, w'_j w') \in \text{Obs}(i, j)$.
- (T2) For all $w \in \langle X \rangle$, we have $\text{o}_{i,j}(\text{Lw}_{\sigma}(g_j)w, 1; 1, w \text{Lw}_{\sigma}(g_i)), \text{o}_{i,j}(1, w \text{Lw}_{\sigma}(g_j); \text{Lw}_{\sigma}(g_i)w, 1) \in \text{Obs}(i, j)$.

Before going on, let us get rid of these two types of trivial obstructions. The following lemma handles trivial obstructions of type (T1).

Lemma 2.6. *If the S -polynomial of $\text{o}_{i,j}(w_i, w'_i; w_j, w'_j) \in \text{Obs}(i, j)$ has a Gröbner representation in terms of G , then, for all $w, w' \in \langle X \rangle$, the S -polynomial of $\text{o}_{i,j}(w w_i, w'_i w'; w w_j, w'_j w')$ also has a Gröbner representation in terms of G .*

Proof. Without loss of generality, we assume that $S_{i,j}(w_i, w'_i; w_j, w'_j)$ is non-zero. We write $S_{i,j}(w_i, w'_i; w_j, w'_j) = \sum_{k=1}^{\mu} c_k w_k g_{i_k} w'_k$, where $c_k \in K \setminus \{0\}$, $w_k, w'_k \in \langle X \rangle$, and $g_{i_k} \in G$ such that $\text{Lw}_{\sigma}(w_j g_j w'_j) >_{\sigma} \text{Lw}_{\sigma}(w_k g_{i_k} w'_k)$ for all $k \in \{1, \dots, \mu\}$. For all $w, w' \in \langle X \rangle$, it is clear that $S_{i,j}(w w_i, w'_i w'; w w_j, w'_j w') = \sum_{k=1}^{\mu} c_k w w_k g_{i_k} w'_k w'$. Since the word ordering σ is compatible with multiplication, we have $w \text{Lw}_{\sigma}(w_j g_j w'_j) w' >_{\sigma} w \text{Lw}_{\sigma}(w_k g_{i_k} w'_k) w'$ for all $k \in \{1, \dots, \mu\}$. Therefore, we have $\text{Lw}_{\sigma}(w w_j g_j w'_j w') >_{\sigma} \text{Lw}_{\sigma}(w w_k g_{i_k} w'_k w')$ for all $k \in \{1, \dots, \mu\}$ and $S_{i,j}(w w_i, w'_i w'; w w_j, w'_j w') = \sum_{k=1}^{\mu} c_k w w_k g_{i_k} w'_k w'$ is a Gröbner representation in terms of G . □

To deal with trivial obstructions of type (T2), we introduce some terminology as follows.

Definition 2.7. Let $G = \{g_1, \dots, g_s\} \subseteq K\langle X \rangle \setminus \{0\}$ with $s \geq 1$.

- (a) Let $w_1, w_2 \in \langle X \rangle$ be two words. If there exist some words $w, w', w'' \in \langle X \rangle$ and $w \neq 1$ such that $w_1 = w'w$ and $w_2 = ww''$, or $w_1 = ww'$ and $w_2 = w''w$, or $w_1 = w$ and $w_2 = w'ww''$, or $w_1 = w'ww''$ and $w_2 = w$, then we say w_1 and w_2 have an *overlap* at w . Otherwise, we say that w_1 and w_2 have *no overlap*.
- (b) Let $\text{o}_{i,j}(w_i, w'_i; w_j, w'_j) \in \text{Obs}(i, j)$ be an obstruction. If $\text{Lw}_\sigma(g_i)$ and $\text{Lw}_\sigma(g_j)$ have an overlap at $w \in \langle X \rangle \setminus \{1\}$ and if w is a subword of $w_i \text{Lw}_\sigma(g_i) w'_i$, then we say that $\text{o}_{i,j}(w_i, w'_i; w_j, w'_j)$ has an *overlap* at w . Otherwise, we say that $\text{o}_{i,j}(w_i, w'_i; w_j, w'_j)$ has *no overlap*.

Thus, as shown in (T2), there are infinitely many obstructions without overlaps in each $\text{Obs}(i, j)$. The following lemma gets rid of these trivial obstructions.

Lemma 2.8. *If $\text{o}_{i,j}(w_i, w'_i; w_j, w'_j) \in \text{Obs}(i, j)$ has no overlap, then $S_{i,j}(w_i, w'_i; w_j, w'_j)$ has a Gröbner representation in terms of G .*

Proof. See [12], Lemma 5.4. □

Observe that Lemma 2.8 is indeed a non-commutative version of the *product criterion* (or *criterion 2*) of Buchberger (cf. [4]).

Definition 2.9. Let $G = \{g_1, \dots, g_s\} \subseteq K\langle X \rangle \setminus \{0\}$ with $s \geq 1$.

- (a) Let $i, j \in \{1, \dots, s\}$ and $i < j$. An obstruction in $\text{Obs}(i, j)$ is called *non-trivial* if it has an overlap and is of the form $\text{o}_{i,j}(w_i, 1; 1, w'_j)$, or $\text{o}_{i,j}(1, w'_i; w_j, 1)$, or $\text{o}_{i,j}(w_i, w'_i; 1, 1)$, or $\text{o}_{i,j}(1, 1; w_j, w'_j)$ with $w_i, w'_i, w_j, w'_j \in \langle X \rangle$.
- (b) Let $i \in \{1, \dots, s\}$. A self obstruction in $\text{Obs}(i, i)$ is called *non-trivial* if it has an overlap and is of the form $\text{o}_{i,i}(1, w'_i; w_i, 1)$ with $w_i, w'_i \in \langle X \rangle \setminus \{1\}$.
- (c) Let $i, j \in \{1, \dots, s\}$ and $i \leq j$. The set of all non-trivial obstructions of g_i and g_j will be denoted by $\text{NTObs}(i, j)$.

In the literature, a non-trivial obstruction of the form $\text{o}_{i,j}(w_i, 1; 1, w'_j)$ is called a *left obstruction*, a non-trivial obstruction of the form $\text{o}_{i,j}(1, w'_i; w_j, 1)$ is called a *right obstruction*, and a non-trivial obstruction of the form $\text{o}_{i,j}(w_i, w'_i; 1, 1)$ or $\text{o}_{i,j}(1, 1; w_j, w'_j)$ is called a *center obstruction*. We picture four types of obstructions as follows.

w_i	$\text{Lw}_\sigma(g_i)$			$\text{Lw}_\sigma(g_i)$	w'_i
$\text{Lw}_\sigma(g_j)$		w'_j		w_j	$\text{Lw}_\sigma(g_j)$
left obstruction			right obstruction		
w_i	$\text{Lw}_\sigma(g_i)$	w'_i		$\text{Lw}_\sigma(g_i)$	
$\text{Lw}_\sigma(g_j)$			w_j	$\text{Lw}_\sigma(g_j)$	w'_j
center obstruction			center obstruction		

At this point we can refine characterization of Gröbner bases as in Proposition 2.5 in the following way.

Proposition 2.10. (Buchberger Criterion)

Let $G = \{g_1, \dots, g_s\} \subseteq K\langle X \rangle$ be a set of non-zero polynomials which generate a two-sided ideal $I = \langle G \rangle \subseteq K\langle X \rangle$. Then the set G is a σ -Gröbner basis of I if and only if, for each non-trivial obstruction $\text{o}_{i,j}(w_i, w'_i; w_j, w'_j) \in \bigcup_{1 \leq i \leq j \leq s} \text{NTObs}(i, j)$, its S -polynomial $S_{i,j}(w_i, w'_i; w_j, w'_j)$ has a Gröbner representation in terms of G .

Proof. This follows directly from Proposition 2.5 and Lemmas 2.6 and 2.8. In view of Lemma 2.8, it suffices to consider each obstruction with overlap, which is either a non-trivial obstruction or a multiple of a non-trivial obstruction. Further, Lemma 2.6 treats a multiple of a non-trivial obstruction via the corresponding non-trivial obstruction. Therefore, it is sufficient to consider only non-trivial obstructions. \square

This proposition enables us to formulate the following procedure for computing Gröbner bases of two-sided ideals. Note that, in the procedure, by a *fair strategy* we mean a selection strategy which ensures that every obstruction is selected eventually. Since these Gröbner bases need not be finite, we have to content ourselves with an enumerating procedure.

Theorem 2.11. (The Buchberger Procedure)

Let $G = \{g_1, \dots, g_s\} \subseteq K\langle X \rangle$ be a set of non-zero polynomials which generate a two-sided ideal $I = \langle G \rangle \subseteq K\langle X \rangle$. Consider the following sequence of instructions.

- (B1) Let $s' = s$ and $B = \bigcup_{1 \leq i \leq j \leq s'} \text{NTObs}(i, j)$.
- (B2) If $B = \emptyset$, return the result G . Otherwise, select an obstruction $\text{o}_{i,j}(w_i, w'_i; w_j, w'_j) \in B$ using a fair strategy and delete it from B .
- (B3) Compute the S -polynomial $S = S_{i,j}(w_i, w'_i; w_j, w'_j)$ and its normal remainder $S' = \text{NR}_{\sigma, G}(S)$. If $S' = 0$, continue with step (B2).
- (B4) Increase s' by one, append $g_{s'} = S'$ to the set G , and append the set of obstructions $\bigcup_{1 \leq i \leq s'} \text{NTObs}(i, s')$ to the set B . Then continue with step (B2).

This is a procedure that enumerates a σ -Gröbner basis G of I . If I has a finite σ -Gröbner basis, it stops after finitely many steps and the resulting set G is a finite σ -Gröbner basis of I .

Proof. Note that this is a straightforward generalization of the commutative version of Buchberger's algorithm to the non-commutative case. We refer to [12] for the original form of this procedure and to [15], Theorem 4.1.14 for a detailed proof. \square

3 Non-Commutative Gebauer-Möller Criteria

In this section we present non-commutative Gebauer-Möller criteria, which check if an obstruction can be represented by “smaller” obstructions. If so, we declare such obstructions to be *unnecessary*. Before going into details, we choose a well-ordering τ on $\mathbb{T}(F_{s'})$ and use it to order obstructions. In the following, let $s' \geq 1$ and let $G = \{g_1, \dots, g_{s'}\} \subseteq K\langle X \rangle \setminus \{0\}$ be a set of non-commutative polynomials.

Definition 3.1. Let us define a relation τ on $\mathbb{T}(F_{s'})$ as follows. For two terms $w_1 e_i w'_1, w_2 e_j w'_2 \in \mathbb{T}(F_{s'})$, we let $w_1 e_i w'_1 \geq_\tau w_2 e_j w'_2$ if

- (a) $w_1 \text{Lw}_\sigma(g_i) w'_1 >_\sigma w_2 \text{Lw}_\sigma(g_j) w'_2$, or
- (b) $w_1 \text{Lw}_\sigma(g_i) w'_1 = w_2 \text{Lw}_\sigma(g_j) w'_2$ and $i > j$, or
- (c) $w_1 \text{Lw}_\sigma(g_i) w'_1 = w_2 \text{Lw}_\sigma(g_j) w'_2$ and $i = j$ and $w_1 \geq_\sigma w_2$.

One can check that τ is a well-ordering and is compatible with scalar multiplication. The relation τ is called the *module term ordering induced by (σ, G)* on $\mathbb{T}(F_{s'})$.

By definition, for every obstruction $\text{o}_{i,j}(w_i, w'_i; w_j, w'_j) \in \bigcup_{1 \leq i \leq j \leq s'} \text{Obs}(i, j)$, we have $w_i e_i w'_i <_\tau w_j e_j w'_j$. We extend the ordering τ to the set of obstructions $\bigcup_{1 \leq i \leq j \leq s'} \text{Obs}(i, j)$ by committing the following slight abuse of notation.

Definition 3.2. Let τ be the module term ordering induced by (σ, G) on $\mathbb{T}(F_{s'})$. Let $\text{o}_{i,j}(w_i, w'_i; w_j, w'_j), \text{o}_{k,l}(w_k, w'_k; w_l, w'_l)$ be two obstructions in the set $\bigcup_{1 \leq i \leq j \leq s'} \text{Obs}(i, j)$. If we have $w_j e_j w'_j >_\tau w_l e_l w'_l$, or if we have $w_j e_j w'_j = w_l e_l w'_l$ and $w_i e_i w'_i \geq_\tau w_k e_k w'_k$, then we let $\text{o}_{i,j}(w_i, w'_i; w_j, w'_j) \geq_\tau \text{o}_{k,l}(w_k, w'_k; w_l, w'_l)$. The ordering τ is called the ordering *induced by (σ, G)* on the set of obstructions.

One can verify that τ is also a well-ordering on $\bigcup_{1 \leq i \leq j \leq s'} \text{Obs}(i, j)$ and compatible with scalar multiplication.

Now we are ready to generalize the commutative Gebauer-Möller criteria (see [5] and [7]) to the non-commutative case. Recall that, in step (B4) of the Buchberger Procedure, when a new generator $g_{s'}$ is added, we immediately construct new obstructions $\bigcup_{1 \leq i \leq s'} \text{NTObs}(i, s')$. We want to detect unnecessary

obstructions in the set $\bigcup_{1 \leq i \leq s'} \text{NTObs}(i, s')$ of newly constructed obstructions and unnecessary obstructions in the set $\bigcup_{1 \leq i \leq j \leq s'-1} \text{NTObs}(i, j)$ of previously constructed obstructions. We achieve this goal via the following three steps. Firstly, we detect unnecessary obstructions in the set $\bigcup_{1 \leq i \leq s'} \text{NTObs}(i, s')$ with the aid of other obstructions also in this set. Secondly, we detect unnecessary obstructions in the set $\bigcup_{1 \leq i \leq s'} \text{NTObs}(i, s')$ with the aid of obstructions in the set $\bigcup_{1 \leq i \leq j \leq s'-1} \text{NTObs}(i, j)$. Thirdly, we detect unnecessary obstructions in the set $\bigcup_{1 \leq i \leq j \leq s'-1} \text{NTObs}(i, j)$ with the aid of the new generator $g_{s'}$. Indeed, the first two steps correspond to the commutative Gebauer-Möller criteria M and F , while the last step corresponds to criterion B_k (c.f. [7], Subsection 3.4).

The following three lemmas help us to implement the first step, that is, to detect unnecessary obstructions in the set $\bigcup_{1 \leq i \leq s'} \text{NTObs}(i, s')$ of newly constructed obstructions via other obstructions in this set. Note that this step is called a *head reduction step* in [5].

Lemma 3.3. *Let $\text{o}_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i})$, $\text{o}_{j,s'}(w_j, w'_j; w_{s'j}, w'_{s'j})$ be two distinct non-trivial obstructions in $\bigcup_{1 \leq i \leq s'} \text{NTObs}(i, s')$ with two words $w, w' \in \langle X \rangle$ satisfying $w_{s'i} = ww_{s'j}$ and $w'_{s'i} = w'_{s'j}w'$. If $i < j$ and $ww' \neq 1$, then we have*

$$\text{o}_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i}) = w \text{o}_{j,s'}(w_j, w'_j; w_{s'j}, w'_{s'j}) w' + \text{o}_{i,j}(w_i, w'_i; ww_j, w'_j w')$$

with $\text{o}_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i}) >_{\tau} \text{o}_{j,s'}(w_j, w'_j; w_{s'j}, w'_{s'j})$ and $\text{o}_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i}) >_{\tau} \text{o}_{i,j}(w_i, w'_i; ww_j, w'_j w')$. Moreover, if the S -polynomials $S_{j,s'}(w_j, w'_j; w_{s'j}, w'_{s'j})$ and $S_{i,j}(w_i, w'_i; ww_j, w'_j w')$ have Gröbner representations in terms of G , then so does $S_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i})$.

Proof. The equation follows from Definition 2.4.a and from the conditions $w_{s'i} = ww_{s'j}$, $w'_{s'i} = w'_{s'j}w'$ and $i < j$. Because of $ww' > 1$, we have $w_{s'i} \text{Lw}(g_{s'})w'_{s'i} = ww_{s'j} \text{Lw}(g_{s'})w'_{s'j}w' >_{\sigma} w_{s'j} \text{Lw}(g_{s'})w'_{s'j}$. Thus we have $w_{s'i}e_{s'}w'_{s'i} >_{\tau} w_{s'j}e_{s'}w'_{s'j}$ and $\text{o}_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i}) >_{\tau} \text{o}_{j,s'}(w_j, w'_j; w_{s'j}, w'_{s'j})$. In view of $w_{s'i} \text{Lw}(g_{s'})w'_{s'i} = w_i \text{Lw}(g_i)w'_i = ww_j \text{Lw}(g_j)w'_j w'$ and $s' > j$, we get $w_{s'i}e_{s'}w'_{s'i} >_{\tau} ww_j e_j w'_j w'$ and $\text{o}_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i}) >_{\tau} \text{o}_{i,j}(w_i, w'_i; ww_j, w'_j w')$.

Next we show that, if $S_{j,s'}(w_j, w'_j; w_{s'j}, w'_{s'j})$ and $S_{i,j}(w_i, w'_i; ww_j, w'_j w')$ have Gröbner representations in terms of G , then so does $S_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i})$. Clearly we have

$$S_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i}) = w S_{j,s'}(w_j, w'_j; w_{s'j}, w'_{s'j}) w' + S_{i,j}(w_i, w'_i; ww_j, w'_j w').$$

Without loss of generality, we assume that $S_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i})$, $S_{j,s'}(w_j, w'_j; w_{s'j}, w'_{s'j})$ and $S_{i,j}(w_i, w'_i; ww_j, w'_j w')$ are non-zero. Since there is a Gröbner representation for $S_{j,s'}(w_j, w'_j; w_{s'j}, w'_{s'j})$, we have

$$S_{j,s'}(w_j, w'_j; w_{s'j}, w'_{s'j}) = \sum_{k=1}^{\mu} a_k w_k g_{i_k} w'_k$$

with $a_k \in K \setminus \{0\}$, $w_k, w'_k \in \langle X \rangle$, $g_{i_k} \in G$ such that $\text{Lw}_{\sigma}(w_{s'j} g_{s'} w'_{s'j}) >_{\sigma} \text{Lw}_{\sigma}(a_k w_k g_{i_k} w'_k)$ for all $k \in \{1, \dots, \mu\}$. Similarly, for $S_{i,j}(w_i, w'_i; ww_j, w'_j w')$ we

have

$$S_{i,j}(w_i, w'_i; ww_j, w'_j w') = \sum_{l=1}^{\nu} b_l w_l g_{i_l} w'_l$$

with $b_l \in K \setminus \{0\}$, $w_l, w'_l \in \langle X \rangle$, $g_{i_l} \in G$ such that $\text{Lw}_{\sigma}(ww_j g_j w'_j w') >_{\sigma} \text{Lw}_{\sigma}(b_l w_l g_{i_l} w'_l)$ for all $l \in \{1, \dots, \nu\}$. Therefore we have

$$\begin{aligned} S_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i}) &= w \left(\sum_{k=1}^{\mu} a_k w_k g_{i_k} w'_k \right) w' + \sum_{l=1}^{\nu} b_l w_l g_{i_l} w'_l \\ &= \sum_{k=1}^{\mu} a_k w w_k g_{i_k} w'_k w' + \sum_{l=1}^{\nu} b_l w_l g_{i_l} w'_l. \end{aligned}$$

From $w_{s'i} \text{Lw}_{\sigma}(g_{s'}) w'_{s'i} = w w_{s'j} \text{Lw}_{\sigma}(g_{s'}) w'_{s'j} w'$, it follows that $\text{Lw}_{\sigma}(w_{s'i} g_{s'} w'_{s'i}) = \text{Lw}_{\sigma}(w w_{s'j} g_{s'} w'_{s'j} w') >_{\sigma} \text{Lw}_{\sigma}(w w_k g_{i_k} w'_k w')$ for all $k \in \{1, \dots, \mu\}$. By Definition 2.4, we have $\text{Lw}_{\sigma}(w_{s'i} g_{s'} w'_{s'i}) = \text{Lw}_{\sigma}(w_i g_i w'_i) = \text{Lw}_{\sigma}(w w_j g_j w'_j w') >_{\sigma} \text{Lw}_{\sigma}(b_l w_l g_{i_l} w'_l)$ for all $l \in \{1, \dots, \nu\}$. Therefore

$$S_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i}) = \sum_{k=1}^{\mu} a_k w w_k g_{i_k} w'_k w' + \sum_{l=1}^{\nu} b_l w_l g_{i_l} w'_l$$

is a Gröbner representation of $S_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i})$. \square

The following example shows that the obstruction $\text{o}_{i,j}(w_i, w'_i; ww_j, w'_j w')$ in the equation of Lemma 3.3 can be a non-trivial obstruction, i.e. a multiple of a non-trivial obstruction or an obstruction without overlap. Similar phenomena occur in Lemmas 3.5, 3.6, 3.9 and 3.11, as well.

Example 3.4. Consider polynomials $G = \{g_1, g_2, g_3\}$ in the non-commutative polynomial ring $K\langle x, y \rangle$.

- (a) Assume that $\text{Lm}_{\sigma}(g_1) = y^3$, $\text{Lm}_{\sigma}(g_2) = x^2 y^2$ and $\text{Lm}_{\sigma}(g_3) = xyx^2 y$. Then we have $\text{o}_{1,3}(xyx^2, 1; 1, y^2)$, $\text{o}_{2,3}(xy, 1; 1, y) \in \bigcup_{1 \leq i \leq 3} \text{NTObs}(i, 3)$, and

$$\text{o}_{1,3}(xyx^2, 1; 1, y^2) = \text{o}_{2,3}(xy, 1; 1, y)y + \text{o}_{1,2}(xyx^2, 1; xy, y).$$

Observe that $\text{o}_{1,2}(xyx^2, xy; y) = xy \text{o}_{1,2}(x^2, 1; 1, y)$ is a multiple of non-trivial obstruction $\text{o}_{1,2}(x^2, 1; 1, y)$.

- (b) Now assume that $\text{Lm}_{\sigma}(g_1) = (xy)^2$, $\text{Lm}_{\sigma}(g_2) = y$ and $\text{Lm}_{\sigma}(g_3) = xyx^2 y$. Then we have $\text{o}_{1,3}(xyx, 1; 1, xy)$, $\text{o}_{2,3}(x, x^2 y; 1, 1) \in \bigcup_{1 \leq i \leq 3} \text{NTObs}(i, 3)$, and

$$\text{o}_{1,3}(xyx, 1; 1, xy) = \text{o}_{2,3}(x, x^2 y; 1, 1)xy + \text{o}_{1,2}(xyx, 1; x, x^2 yxy).$$

One can check that $\text{o}_{1,2}(xyx, 1; x, x^2 yxy)$ is an obstruction without overlap.

Lemma 3.5. Let $\mathfrak{o}_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i}), \mathfrak{o}_{j,s'}(w_j, w'_j; w_{s'j}, w'_{s'j})$ be two distinct non-trivial obstructions in $\bigcup_{1 \leq i \leq s'} \text{NTObs}(i, s')$ with two words $w, w' \in \langle X \rangle$ satisfying $w_{s'i} = ww_{s'j}$ and $w'_{s'i} = w'_{s'j}w'$. If $i > j$, then we have

$$\mathfrak{o}_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i}) = w\mathfrak{o}_{j,s'}(w_j, w'_j; w_{s'j}, w'_{s'j})w' - \mathfrak{o}_{j,i}(ww_j, w'_jw'; w_i, w'_i)$$

with $\mathfrak{o}_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i}) >_{\tau} \mathfrak{o}_{j,s'}(w_j, w'_j; w_{s'j}, w'_{s'j})$ and $\mathfrak{o}_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i}) >_{\tau} \mathfrak{o}_{j,i}(ww_j, w'_jw'; w_i, w'_i)$. Moreover, if the S -polynomials $S_{j,s'}(w_j, w'_j; w_{s'j}, w'_{s'j})$ and $S_{j,i}(ww_j, w'_jw'; w_i, w'_i)$ have Gröbner representations in terms of G , then so does $S_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i})$.

Proof. The equation follows from Definition 2.4.a and from the conditions $w_{s'i} = ww_{s'j}$, $w'_{s'i} = w'_{s'j}w'$ and $i > j$. From $w_{s'i} \text{Lw}(g_{s'})w'_{s'i} = ww_{s'j} \text{Lw}(g_{s'})w'_{s'j}w' \geq_{\sigma} w_{s'j} \text{Lw}(g_{s'})w'_{s'j}$ and $i > j$, it follows that $w_{s'i}e_{s'}w'_{s'i} \geq_{\tau} w_{s'j}e_{s'}w'_{s'j}$ and $w_i e_i w'_i >_{\tau} w_j e_j w'_j$. Thus we have $\mathfrak{o}_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i}) >_{\tau} \mathfrak{o}_{j,s'}(w_j, w'_j; w_{s'j}, w'_{s'j})$. Because of $w_{s'i} \text{Lw}(g_{s'})w'_{s'i} = w_i \text{Lw}(g_i)w'_i$ and $s' > i$, we have $w_{s'i}e_{s'}w'_{s'i} >_{\tau} w_i e_i w'_i$ and $\mathfrak{o}_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i}) >_{\tau} \mathfrak{o}_{j,i}(ww_j, w'_jw'; w_i, w'_i)$. The second part can be proved by following the same argument as in the proof of Lemma 3.3. \square

Lemma 3.6. Let $\mathfrak{o}_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i}), \mathfrak{o}_{i,s'}(w_j, w'_j; w_{s'j}, w'_{s'j}) \in \text{NTObs}(i, s')$ be two distinct non-trivial obstructions with two words $w, w' \in \langle X \rangle$ satisfying $w_{s'i} = ww_{s'j}$ and $w'_{s'i} = w'_{s'j}w'$. If $ww' \neq 1$ or if $ww' = 1$ and $w_i >_{\sigma} w_j$, then we have

$$\mathfrak{o}_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i}) = w\mathfrak{o}_{i,s'}(w_j, w'_j; w_{s'j}, w'_{s'j})w' + \mathfrak{o}_{i,i}(w_i, w'_i; ww_j, w'_jw')$$

with $\mathfrak{o}_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i}) >_{\tau} \mathfrak{o}_{i,s'}(w_j, w'_j; w_{s'j}, w'_{s'j})$ and $\mathfrak{o}_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i}) >_{\tau} \mathfrak{o}_{i,i}(w_i, w'_i; ww_j, w'_jw')$. Moreover, if the S -polynomials $S_{i,s'}(w_j, w'_j; w_{s'j}, w'_{s'j})$ and $S_{i,i}(w_i, w'_i; ww_j, w'_jw')$ have Gröbner representations in terms of G , then so does $\mathfrak{o}_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i})$.

Proof. The equation follows from Definition 2.4.a and from the conditions $w_{s'i} = ww_{s'j}$ and $w'_{s'i} = w'_{s'j}w'$. From $w_{s'i} \text{Lw}(g_{s'})w'_{s'i} = ww_{s'j} \text{Lw}(g_{s'})w'_{s'j}w'$ and the conditions $ww' \neq 1$ or $ww' = 1$ and $w_i >_{\sigma} w_j$, it follows that $w_{s'i}e_{s'}w'_{s'i} >_{\tau} w_{s'j}e_{s'}w'_{s'j}$ and $\mathfrak{o}_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i}) >_{\tau} \mathfrak{o}_{i,s'}(w_j, w'_j; w_{s'j}, w'_{s'j})$. And from $w_{s'i} \text{Lw}(g_{s'})w'_{s'i} = w_i \text{Lw}(g_i)w'_i = ww_j \text{Lw}(g_i)w'_jw'$ and $s' > i$, it follows that $w_{s'i}e_{s'}w'_{s'i} >_{\tau} ww_j e_i w'_j w'$ and $\mathfrak{o}_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i}) >_{\tau} \mathfrak{o}_{i,i}(w_i, w'_i; ww_j, w'_jw')$. One can verify the second part by following the same approach as in the proof of Lemma 3.3. \square

In the following, we present the non-commutative multiply criterion and the leading word criterion. They are non-commutative analogues of the Gebauer-Möller criteria M and F, respectively.

Proposition 3.7. (Non-Commutative Multiply Criterion)

Suppose that $\mathfrak{o}_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i})$ and $\mathfrak{o}_{j,s'}(w_j, w'_j; w_{s'j}, w'_{s'j})$ are two distinct non-trivial obstructions in $\bigcup_{1 \leq i \leq s'} \text{NTObs}(i, s')$ such that there exist two words $w, w' \in \langle X \rangle$ satisfying $w_{s'i} = ww_{s'j}$ and $w'_{s'i} = w'_{s'j}w'$. Then we can remove

$\mathfrak{o}_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i})$ from $\bigcup_{1 \leq i \leq s'} \text{NTObs}(i, s')$ in the execution of the Buchberger Procedure if $ww' \neq 1$.

Proof. According to the previous lemmas, the obstruction $\mathfrak{o}_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i})$ can be represented as

$$\mathfrak{o}_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i}) = w\mathfrak{o}_{j,s'}(w_j, w'_j; w_{s'j}, w'_{s'j})w' + a\mathfrak{o}_{k,l}(w_k, w'_k; w_l, w'_l)$$

with $a \in \{1, -1\}$ and $k = \min\{i, j\}, l = \max\{i, j\}$. To prove that $\mathfrak{o}_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i})$ is strictly larger than $\mathfrak{o}_{j,s'}(w_j, w'_j; w_{s'j}, w'_{s'j})$ and $\mathfrak{o}_{k,l}(w_k, w'_k; w_l, w'_l)$, we consider two cases. If $i > j$, then the result follows from Lemma 3.5; if $i \leq j$, then the result follows from Lemmas 3.3 and 3.6. Moreover, $S_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i})$ has a Gröbner representation in terms of G if $S_{j,s'}(w_j, w'_j; w_{s'j}, w'_{s'j})$ and $S_{k,l}(w_k, w'_k; w_l, w'_l)$ have Gröbner representations in terms of G . Theorem 2.11 ensures that $S_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i})$ has a Gröbner representation in terms of G . Note that the obstruction $\mathfrak{o}_{k,l}(w_k, w'_k; w_l, w'_l)$ can be either a multiple of a non-trivial obstruction or an obstruction without overlap (for instance, see Example 3.4). If $\mathfrak{o}_{k,l}(w_k, w'_k; w_l, w'_l)$ is a multiple of a non-trivial obstruction, then Lemma 2.6 and Theorem 2.11 guarantee that $S_{k,l}(w_k, w'_k; w_l, w'_l)$ has a Gröbner representation in terms of G . If $\mathfrak{o}_{k,l}(w_k, w'_k; w_l, w'_l)$ is an obstruction without overlap, then, by Lemma 2.8, its S-polynomial has a Gröbner representation in terms of G . Now the conclusion follows from Proposition 2.10 and Theorem 2.11. \square

Proposition 3.8. (Non-Commutative Leading Word Criterion)

Suppose that $\mathfrak{o}_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i})$ and $\mathfrak{o}_{j,s'}(w_j, w'_j; w_{s'j}, w'_{s'j})$ are two distinct non-trivial obstructions in $\bigcup_{1 \leq i \leq s'} \text{NTObs}(i, s')$ such that there exist two words $w, w' \in \langle X \rangle$ satisfying $w_{s'i} = ww_{s'j}$ and $w'_{s'i} = w'_{s'j}w'$. Then $\mathfrak{o}_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i})$ can be removed from $\bigcup_{1 \leq i \leq s'} \text{NTObs}(i, s')$ in the execution of the Buchberger Procedure if one of the following conditions is satisfied.

- (a) $i > j$.
- (b) $i = j$ and $ww' = 1$ and $w_i >_\sigma w_j$.

Proof. Observe that condition (a) corresponds to Lemma 3.5, while condition (b) corresponds to Lemma 3.6. We represent $\mathfrak{o}_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i})$ as

$$\mathfrak{o}_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i}) = w\mathfrak{o}_{j,s'}(w_j, w'_j; w_{s'j}, w'_{s'j})w' - \mathfrak{o}_{j,i}(ww_j, w'_jw'; w_i, w'_i).$$

By Lemmas 3.5 and 3.6, we have $\mathfrak{o}_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i})$ is strictly larger than $\mathfrak{o}_{j,s'}(w_j, w'_j; w_{s'j}, w'_{s'j})$ and $\mathfrak{o}_{j,i}(ww_j, w'_jw'; w_i, w'_i)$. Moreover, $S_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i})$ has a Gröbner representation in terms of G if $S_{j,s'}(w_j, w'_j; w_{s'j}, w'_{s'j})$ and $S_{j,i}(ww_j, w'_jw'; w_i, w'_i)$ have Gröbner representations in terms of G . Theorem 2.11 ensures that $S_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i})$ has a Gröbner representation in terms of G . Note that the obstruction $\mathfrak{o}_{j,i}(ww_j, w'_jw'; w_i, w'_i)$ can be either a multiple of a non-trivial obstruction or an obstruction without overlap (for instance, see Example 3.4). If $\mathfrak{o}_{j,i}(ww_j, w'_jw'; w_i, w'_i)$ is a multiple of a non-trivial obstruction, then Lemma 2.6 and Theorem 2.11 guarantee that $S_{j,i}(ww_j, w'_jw'; w_i, w'_i)$

has a Gröbner representation in terms of G . If $\mathbf{o}_{j,i}(ww_j, w'_j w'; w_i, w'_i)$ is an obstruction without overlap, then, by Lemma 2.8, its S-polynomial has a Gröbner representation in terms of G . Now the conclusion follows from Proposition 2.10 and Theorem 2.11. \square

Next we work on detecting unnecessary obstructions in $\bigcup_{1 \leq i \leq s'} \text{NTObs}(i, s')$ via obstructions in the set $\bigcup_{1 \leq i \leq j \leq s'-1} \text{NTObs}(i, j)$ of previously constructed obstructions. Note that this step is called a *tail reduction step* in [5].

Lemma 3.9. *Let $\mathbf{o}_{j,s'}(w_{js'}, w'_{js'}; w_{s'}, w'_{s'}) \in \bigcup_{1 \leq i \leq s'} \text{NTObs}(i, s')$ and $\mathbf{o}_{i,j}(w_i, w'_i; w_{ji}, w'_{ji}) \in \bigcup_{1 \leq i \leq j \leq s'-1} \text{NTObs}(i, j)$ be two non-trivial obstructions. If there exist two words $w, w' \in \langle X \rangle$ such that $w_{js'} = ww_{ji}$ and $w'_{js'} = w'_{ji} w'$, then we have*

$$\mathbf{o}_{j,s'}(w_{js'}, w'_{js'}; w_{s'}, w'_{s'}) = -w \mathbf{o}_{i,j}(w_i, w'_i; w_{ji}, w'_{ji}) w' + \mathbf{o}_{i,s'}(ww_i, w'_i w'; w_{s'}, w'_{s'})$$

with $\mathbf{o}_{j,s'}(w_{js'}, w'_{js'}; w_{s'}, w'_{s'}) >_{\tau} \mathbf{o}_{i,j}(w_i, w'_i; w_{ji}, w'_{ji})$ and $\mathbf{o}_{j,s'}(w_{js'}, w'_{js'}; w_{s'}, w'_{s'}) >_{\tau} \mathbf{o}_{i,s'}(ww_i, w'_i w'; w_{s'}, w'_{s'})$. Moreover, if the S-polynomials $S_{i,j}(w_i, w'_i; w_{ji}, w'_{ji})$ and $S_{i,s'}(ww_i, w'_i w'; w_{s'}, w'_{s'})$ have Gröbner representations in terms of G , then so does $S_{j,s'}(w_{js'}, w'_{js'}; w_{s'}, w'_{s'})$.

Proof. The equation follows from Definition 2.4.a and from the conditions $w_{js'} = ww_{ji}$ and $w'_{js'} = w'_{ji} w'$. We have $\mathbf{o}_{j,s'}(w_{js'}, w'_{js'}; w_{s'}, w'_{s'}) >_{\tau} \mathbf{o}_{i,j}(w_i, w'_i; w_{ji}, w'_{ji})$ for $w_{s'} e_{s'} w'_{s'} >_{\tau} w_{js'} e_j w'_{js'} = ww_{ji} e_j w'_{ji} w \geq_{\tau} w_{ji} e_j w'_{ji}$. Because of $w_{js'} e_j w'_{js'} = ww_{ji} e_j w'_{ji} w >_{\tau} ww_i e_i w'_i w'$, we have $\mathbf{o}_{j,s'}(w_{js'}, w'_{js'}; w_{s'}, w'_{s'}) >_{\tau} \mathbf{o}_{i,s'}(ww_i, w'_i w'; w_{s'}, w'_{s'})$. Again, we can prove the second part by following the same argument as in the proof of Lemma 3.3. \square

Note that the obstruction $\mathbf{o}_{i,s'}(ww_i, w'_i w'; w_{s'}, w'_{s'})$ in Lemma 3.9 can be either a multiple of a non-trivial obstruction or an obstruction without overlap. However, it suffices for us to consider only the latter case, since the former case has been considered in Proposition 3.7, and, more precisely, in Lemma 3.5.

Proposition 3.10. (Non-Commutative Tail Reduction)

Suppose that $\mathbf{o}_{j,s'}(w_{js'}, w'_{js'}; w_{s'}, w'_{s'})$ and $\mathbf{o}_{i,j}(w_i, w'_i; w_{ji}, w'_{ji})$ are non-trivial obstructions in $\bigcup_{1 \leq i \leq s'} \text{NTObs}(i, s')$ and $\bigcup_{1 \leq i \leq j \leq s'-1} \text{NTObs}(i, j)$, respectively, such that there exist two words $w, w' \in \langle X \rangle$ satisfying $w_{js'} = ww_{ji}$ and $w'_{js'} = w'_{ji} w'$. If ww_i is a multiple of $w_{s'} \text{Lw}_{\sigma}(g_{s'})$ or if $w'_i w'$ is a multiple of $\text{Lw}_{\sigma}(g_{s'}) w'_{s'}$, then $\mathbf{o}_{j,s'}(w_{js'}, w'_{js'}; w_{s'}, w'_{s'})$ can be removed from $\bigcup_{1 \leq i \leq s'} \text{NTObs}(i, s')$ in the execution of the Buchberger Procedure.

Proof. By Lemma 3.9, the obstruction $\mathbf{o}_{j,s'}(w_{js'}, w'_{js'}; w_{s'}, w'_{s'})$ can be represented as

$$\mathbf{o}_{j,s'}(w_{js'}, w'_{js'}; w_{s'}, w'_{s'}) = -w \mathbf{o}_{i,j}(w_i, w'_i; w_{ji}, w'_{ji}) w' + \mathbf{o}_{i,s'}(ww_i, w'_i w'; w_{s'}, w'_{s'})$$

with $\mathbf{o}_{j,s'}(w_{js'}, w'_{js'}; w_{s'}, w'_{s'}) >_{\tau} \mathbf{o}_{i,j}(w_i, w'_i; w_{ji}, w'_{ji})$ and $\mathbf{o}_{j,s'}(w_{js'}, w'_{js'}; w_{s'}, w'_{s'}) >_{\tau} \mathbf{o}_{i,s'}(ww_i, w'_i w'; w_{s'}, w'_{s'})$. Moreover, if the S-polynomials $S_{i,j}(w_i, w'_i; w_{ji}, w'_{ji})$

and $S_{i,s'}(ww_i, w'_i w'; w_{s'}, w'_{s'})$ have Gröbner representations in terms of G , then so does $S_{j,s'}(w_{js'}, w'_{js'}; w_{s'}, w'_{s'})$. Theorem 2.11 ensures that $S_{i,j}(w_i, w'_i; w_{ji}, w'_{ji})$ has a Gröbner representation in terms of G . Note that ww_i is a multiple of $w_{s'} \text{Lw}_\sigma(g_{s'})$ or $w'_i w'$ is a multiple of $\text{Lw}_\sigma(g_{s'}) w'_{s'}$. This implies that $\text{o}_{i,s'}(ww_i, w'_i w'; w_{s'}, w'_{s'})$ has no overlap. By Lemma 2.8, $S_{i,s'}(ww_i, w'_i w'; w_{s'}, w'_{s'})$ has a Gröbner representation in terms of G . Now the conclusion follows from Proposition 2.10 and Theorem 2.11. \square

Our experiments in the final section show that, after applying the previous two criteria, the Non-Commutative Tail Reduction is unlikely to apply in the Buchberger Procedure. This may be due to the fact that frequently the Non-Commutative Multiply Criterion and the Non-Commutative Leading Word Criterion have already detected all unnecessary obstructions in the set $\bigcup_{1 \leq i \leq s'} \text{NTObs}(i, s')$ of newly constructed obstructions.

So far we have detected unnecessary obstructions in the set $\bigcup_{1 \leq i \leq s'} \text{O}(i, s')$ of newly constructed obstructions. Intuitively, we are also able to detect unnecessary obstructions in the set $\bigcup_{1 \leq i \leq j \leq s'-1} \text{Obs}(i, j)$ of previously constructed obstructions. Thus, in the last step, we detect unnecessary obstructions in this set by using the new generator $g_{s'}$.

Lemma 3.11. *Let $\text{o}_{i,j}(w_i, w'_i; w_j, w'_j) \in \bigcup_{1 \leq i \leq j \leq s'-1} \text{NTObs}(i, j)$ be a non-trivial obstruction. If there are two words $w_{s'}, w'_{s'} \in \langle X \rangle$ satisfying $w_j \text{Lw}_\sigma(g_j) w'_j = w_{s'} \text{Lw}_\sigma(g_{s'}) w'_{s'}$, then we can represent $\text{o}_{i,j}(w_i, w'_i; w_j, w'_j)$ as*

$$\text{o}_{i,j}(w_i, w'_i; w_j, w'_j) = \text{o}_{i,s'}(w_i, w'_i; w_{s'}, w'_{s'}) - \text{o}_{j,s'}(w_j, w'_j; w_{s'}, w'_{s'}).$$

Moreover, if $S_{i,s'}(w_i, w'_i; w_{s'}, w'_{s'})$ and $S_{j,s'}(w_j, w'_j; w_{s'}, w'_{s'})$ have Gröbner representations in terms of G , then so does $S_{i,j}(w_i, w'_i; w_j, w'_j)$.

Proof. The equation follows from Definition 2.4.a and the condition $w_j \text{Lw}_\sigma(g_j) w'_j = w_{s'} \text{Lw}_\sigma(g_{s'}) w'_{s'}$. The proof of the second part is analogous to the proof of the second part of Lemma 3.3. \square

The following example shows that the obstruction $\text{o}_{i,s'}(w_i, w'_i; w_{s'}, w'_{s'})$ in the equation of Lemma 3.11 can be either an obstruction without overlap or a multiple of a non-trivial obstruction. In the case that $\text{o}_{i,s'}(w_i, w'_i; w_{s'}, w'_{s'})$ is a multiple of a non-trivial obstruction, say $\text{o}_{i,s'}(\tilde{w}_i, \tilde{w}'_i; \tilde{w}_{s'}, \tilde{w}'_{s'})$, the example shows that it is not necessary to have $\text{o}_{i,s'}(w_i, w'_i; w_{s'}, w'_{s'}) >_\tau \text{o}_{i,s'}(\tilde{w}_i, \tilde{w}'_i; \tilde{w}_{s'}, \tilde{w}'_{s'})$ (compared to Lemmas 3.3, 3.5, 3.6 and 3.9). The same also holds for the obstruction $\text{o}_{j,s'}(w_j, w'_j; w_{s'}, w'_{s'})$ in the equation of Lemma 3.11.

Example 3.12. Consider polynomials $G = \{g_1, g_2, g_3\}$ in the non-commutative polynomial ring $K\langle x, y \rangle$ with $\text{Lm}_\sigma(g_1) = x^3 y x$, $\text{Lm}_\sigma(g_2) = x^2$ and $\text{Lm}_\sigma(g_3) = x$. We have $\text{o}_{1,2}(1, 1; x, yx) \in \bigcup_{1 \leq i \leq j \leq 2} \text{NTObs}(i, j)$ and $x \text{Lw}_\sigma(g_2) y x = x^3 y x = x^3 y \text{Lw}_\sigma(g_3)$ and

$$\text{o}_{1,2}(1, 1; x, yx) = \text{o}_{1,3}(1, 1; x^3 y, 1) - \text{o}_{2,3}(x, yx; x^3 y, 1).$$

One can check that $\mathfrak{o}_{1,3}(1, 1; x^3y, 1)$ is a non-trivial obstruction in $\text{NTObs}(1, 3)$ and $\mathfrak{o}_{1,2}(1, 1; x, yx) <_{\tau} \mathfrak{o}_{1,3}(1, 1; x^3y, 1)$. Moreover, $\mathfrak{o}_{2,3}(x, yx; x^3y, 1)$ is an obstruction without overlap.

The following is a non-commutative analogue of the Gebauer-Möller criterion B_k , which is also known as the *chain criterion* (or *criterion 1*) of Buchberger (cf. [4]).

Proposition 3.13. (Non-Commutative Backward Criterion)

Suppose that $\mathfrak{o}_{i,j}(w_i, w'_i; w_j, w'_j) \in \bigcup_{1 \leq i \leq j \leq s'-1} \text{NTObs}(i, j)$ is a non-trivial obstruction. Then in the execution of the Buchberger Procedure $\mathfrak{o}_{i,j}(w_i, w'_i; w_j, w'_j)$ can be removed from $\bigcup_{1 \leq i \leq j \leq s'-1} \text{NTObs}(i, j)$ if the following three conditions are satisfied.

- (a) There are $w_{s'}, w'_{s'} \in \langle X \rangle$ such that $w_j \text{Lw}_{\sigma}(g_j)w'_j = w_{s'} \text{Lw}_{\sigma}(g_{s'})w'_{s'}$.
- (b) The obstruction $\mathfrak{o}_{i,s'}(w_i, w'_i; w_{s'}, w'_{s'})$ is either an obstruction without overlap or a multiple of a non-trivial obstruction in $\bigcup_{1 \leq i \leq s'} \text{NTObs}(i, s')$.
- (c) The obstruction $\mathfrak{o}_{j,s'}(w_j, w'_j; w_{s'}, w'_{s'})$ is either an obstruction without overlap or a multiple of a non-trivial obstruction in $\bigcup_{1 \leq i \leq s'} \text{NTObs}(i, s')$.

Proof. By Lemma 3.11, we can represent $\mathfrak{o}_{i,j}(w_i, w'_i; w_j, w'_j)$ as

$$\mathfrak{o}_{i,j}(w_i, w'_i; w_j, w'_j) = \mathfrak{o}_{i,s'}(w_i, w'_i; w_{s'}, w'_{s'}) - \mathfrak{o}_{j,s'}(w_j, w'_j; w_{s'}, w'_{s'}).$$

Moreover, $S_{i,j}(w_i, w'_i; w_j, w'_j)$ has a Gröbner representations in terms of G if $S_{i,s'}(w_i, w'_i; w_{s'}, w'_{s'})$ and $S_{j,s'}(w_j, w'_j; w_{s'}, w'_{s'})$ have Gröbner representations in terms of G . If $\mathfrak{o}_{i,s'}(w_i, w'_i; w_{s'}, w'_{s'})$ is an obstruction without overlap, then, by Lemma 2.8, its S-polynomial has a Gröbner representations in terms of G . If it is a multiple of a non-trivial obstruction in $\bigcup_{1 \leq i \leq s'} \text{NTObs}(i, s')$, then Lemma 2.8 and Theorem 2.11 ensure that $S_{i,s'}(w_i, w'_i; w_{s'}, w'_{s'})$ has a Gröbner representations in terms of G . By the same argument, one can show that $S_{j,s'}(w_j, w'_j; w_{s'}, w'_{s'})$ has a Gröbner representations in terms of G . Now the conclusion follows from Proposition 2.10 and Theorem 2.11. \square

We would like to mention that the Non-Commutative Backward Criterion given in Proposition 3.13 covers in particular all useless obstructions presented by T. Mora in [12], Lemma 5.5.

Remark 3.14. In order to apply Propositions 3.7, 3.10 and 3.13 to remove unnecessary obstructions during the execution of the Buchberger Procedure, it is crucial to make sure that the S-polynomials of those removed obstructions have Gröbner representations.

- (a) Propositions 3.7, 3.8 and 3.10 remove unnecessary non-trivial obstructions, say $\mathfrak{o}_{i,s'}(w_i, w'_i; w_{s'}, w'_{s'})$, from the set $\bigcup_{1 \leq i \leq s'} \text{NTObs}(i, s')$ of newly constructed obstructions. The Gröbner representation of the S-polynomial

$S_{i,s'}(w_i, w'_i; w_{s'}, w'_{s'})$ depends on the Gröbner representations of the S-polynomials of two smaller obstructions in the sets $\bigcup_{1 \leq i \leq j \leq s'-1} \text{Obs}(i, j)$ and $\bigcup_{1 \leq i \leq s'} \text{Obs}(i, s')$.

- (b) Proposition 3.13 removes unnecessary obstructions, say $\text{o}_{i,j}(w_i, w'_i; w_j, w'_j)$, from the set $\bigcup_{1 \leq i \leq j \leq s'-1} \text{Obs}(i, j)$ of previously constructed obstructions. The Gröbner representation of $S_{i,j}(w_i, w'_i; w_j, w'_j)$ depends on the Gröbner representations of the S-polynomials of two obstructions, say $\text{o}_{k,s'}(w_k, w'_k; w_{s'k}, w'_{s'k})$ and $\text{o}_{l,s'}(w_l, w'_l; w_{s'l}, w'_{s'l})$, in $\bigcup_{1 \leq i \leq s'} \text{Obs}(i, s')$, which are not necessarily smaller than $\text{o}_{i,j}(w_i, w'_i; w_j, w'_j)$. If $\text{o}_{k,s'}(w_k, w'_k; w_{s'k}, w'_{s'k})$ is a multiple of a non-trivial obstruction, say $\text{o}_{k,s'}(\tilde{w}_k, \tilde{w}'_k; \tilde{w}_{s'k}, \tilde{w}'_{s'k})$, in $\bigcup_{1 \leq i \leq s'} \text{NTObs}(i, s')$, then, before removing $\text{o}_{i,j}(w_i, w'_i; w_j, w'_j)$, it is important to ensure that $\text{o}_{k,s'}(\tilde{w}_k, \tilde{w}'_k; \tilde{w}_{s'k}, \tilde{w}'_{s'k})$ is in $\bigcup_{1 \leq i \leq s'} \text{NTObs}(i, s')$. The same check should be applied to $\text{o}_{l,s'}(w_l, w'_l; w_{s'l}, w'_{s'l})$.

Observe that Propositions 3.7, 3.8 and 3.13 are actually generalizations of the well-known Gebauer-Möller criteria (see [5] and [7]) in commutative polynomial rings. More precisely, Propositions 3.7, 3.8 and 3.10 correspond to criterion M , criterion F and criterion B_k , respectively (c.f. [7], Subsection 3.4).

Using the Gebauer-Möller criteria, we can improve the Buchberger Procedure as follows.

Theorem 3.15. (Improved Buchberger Procedure)

In the setting of Theorem 2.11, we replace step (B4) by the following sequence of instructions.

- (4a) *Increase s' by one. Append $g_{s'} = S'$ to the set G , and form the set of non-trivial obstructions $\text{NTObs}(s') = \bigcup_{1 \leq i \leq s'} \text{NTObs}(i, s')$.*
- (4b) *Remove from $\text{NTObs}(s')$ all obstructions $\text{o}_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i})$ such that there exists an obstruction $\text{o}_{j,s'}(w_j, w'_j; w_{s'j}, w'_{s'j}) \in \text{NTObs}(s')$ with the properties that there exist two words $w, w' \in \langle X \rangle$ satisfying $w_{s'i} = ww_{s'j}$, $w'_{s'i} = w'_{s'j}w'$ and $ww' \neq 1$.*
- (4c) *Remove from $\text{NTObs}(s')$ all obstructions $\text{o}_{i,s'}(w_i, w'_i; w_{s'i}, w'_{s'i})$ such that there exists an obstruction $\text{o}_{j,s'}(w_j, w'_j; w_{s'j}, w'_{s'j}) \in \text{NTObs}(s')$ with the properties that there exist two words $w, w' \in \langle X \rangle$ satisfying $w_{s'i} = ww_{s'j}$, $w'_{s'i} = w'_{s'j}w'$, and such that $i > j$, or $i = j$ and $ww' = 1$ and $w_i >_\sigma w_j$.*
- (4d) *Remove from $\text{NTObs}(s')$ all obstructions $\text{o}_{j,s'}(w_{js'}, w'_{js'}; w_{s'}, w'_{s'})$ such that there exists an obstruction $\text{o}_{i,j}(w_i, w'_i; w_{ji}, w'_{ji}) \in B$ with the properties that there exist two words $w, w' \in \langle X \rangle$ satisfying $w_{js'} = ww_{ji}$, $w'_{js'} = w'_{ji}w'$, and such that $\text{o}_{i,s'}(ww_i, w'_i w'; w_{s'}, w'_{s'})$ has no overlap.*
- (4e) *Remove from B all obstructions $\text{o}_{i,j}(w_i, w'_i; w_j, w'_j)$ such that there exist two words $w, w' \in \langle X \rangle$ satisfying $w \text{Lw}_\sigma(g_{s'})w' = w_j \text{Lw}_\sigma(g_j)w'_j$, and such that the following conditions are satisfied.*

- (i) $\circ_{i,s'}(w_i, w'_i; w_{s'}, w'_{s'})$ is either an obstruction without overlap or a multiple of a non-trivial obstruction in $\text{NTObs}(s')$.
- (ii) $\circ_{j,s'}(w_j, w'_j; w_{s'}, w'_{s'})$ is either an obstruction without overlap or a multiple of a non-trivial obstruction in $\text{NTObs}(s')$.

(4f) Replace B by $B \cup \text{NTObs}(s')$ and continue with step (B2).

Then the resulting set of instructions is a procedure that enumerates a σ -Gröbner basis G of I . If I has a finite σ -Gröber basis, it stops after finitely many steps and the resulting set G is a finite σ -Gröbner basis of I .

Proof. This follows from Theorem 2.11 and Propositions 3.7, 3.8, 3.10 and 3.13. \square

4 Experiments and Conclusions

In this section we want to present some experimental data which illustrate the performance of the Gebauer-Möller criteria presented in Propositions 3.7, 3.10 and 3.13. The computations are based on an implementation (using C++) in an experimental version of the ApCoCoA library (see [1]) by the second author.

Example 4.1. Consider the non-commutative polynomial ring $\mathbb{Q}\langle a, b \rangle$ equipped with the word ordering LLex on $\langle a, b \rangle$ such that $a >_{\text{LLex}} b$. We take the list of *finite generalized triangle groups* from [13], Theorem 2.12 and construct a list of ideals in $\mathbb{Q}\langle a, b \rangle$. For $k = 1, \dots, 13$ let $I_k = \langle G_k \rangle \subseteq \mathbb{Q}\langle a, b \rangle$ be the ideal generated by the following set of polynomials $G_k \subseteq \mathbb{Q}\langle a, b \rangle$.

$$\begin{aligned}
G_1 &= \{a^2 - 1, b^3 - 1, (ababab^2ab^2)^2 - 1\}, \\
G_2 &= \{a^2 - 1, b^3 - 1, (ababab^3)^3 - 1\}, \\
G_3 &= \{a^3 - 1, b^3 - 1, (abab^2)^2 - 1\}, \\
G_4 &= \{a^3 - 1, b^3 - 1, (aba^2b^2)^2 - 1\}, \\
G_5 &= \{a^2 - 1, b^5 - 1, (abab^2)^2 - 1\}, \\
G_6 &= \{a^2 - 1, b^5 - 1, (ababab^4)^2 - 1\}, \\
G_7 &= \{a^2 - 1, b^5 - 1, (abab^2ab^4)^2 - 1\}, \\
G_8 &= \{a^2 - 1, b^4 - 1, (ababab^3)^2 - 1\}, \\
G_9 &= \{a^2 - 1, b^3 - 1, (abab^2)^2 - 1\}, \\
G_{10} &= \{a^2 - 1, b^3 - 1, (ababab^2)^2 - 1\}, \\
G_{11} &= \{a^2 - 1, b^3 - 1, (abababab^2)^2 - 1\}, \\
G_{12} &= \{a^2 - 1, b^3 - 1, (ababab^2abab^2)^2 - 1\}, \\
G_{13} &= \{a^2 - 1, b^3 - 1, (ababababab^2ab^2)^2 - 1\}.
\end{aligned}$$

The following table lists some numbers of polynomials and obstructions treated by the Improved Buchberger Procedure given in Theorem 3.15.

k	$\#(Gb)$	$\#(RedGb)$	$\#(TolObs)$	$\#(SelObs)$	$\#(M)$	$\#(F)$	$\#(TR)$	$\#(B_k)$	ρ
1	60	35	6592	247	6077	45	0	223	0.0375
2	131	96	30771	530	29683	69	0	489	0.0172
3	49	40	2721	194	2388	9	0	130	0.0713
4	66	28	5047	262	4501	43	0	241	0.0519
5	36	21	1686	119	1443	23	0	101	0.0706
6	199	164	51077	880	48962	32	0	1203	0.0172
7	199	164	51285	878	49175	19	0	1213	0.0171
8	52	37	3602	190	3195	21	0	196	0.0528
9	11	5	150	31	98	8	0	13	0.2067
10	22	15	741	75	606	18	0	42	0.1012
11	30	21	1573	117	1322	51	0	83	0.0744
12	96	70	16495	365	15648	93	0	389	0.0221
13	220	194	87507	1021	84903	149	0	1434	0.0117

Here we used the following abbreviations.

- $\#(Gb)$ is the number of elements of the Gröbner basis returned by the procedure.
- $\#(RedGb)$ is the cardinality of the reduced Gröbner basis of the corresponding ideal.
- $\#(TolObs)$ is the total number of non-trivial obstructions.
- $\#(SelObs)$ is the number of actually selected non-trivial obstructions.
- $\#(M)$ is the number of unnecessary non-trivial obstructions detected by the Non-Commutative Multiply Criterion given in Proposition 3.7.
- $\#(F)$ is the number of unnecessary non-trivial obstructions detected by the Non-Commutative Leading Word Criterion given in Proposition 3.8.
- $\#(TR)$ is the number of unnecessary non-trivial obstructions detected by the Non-Commutative Tail Reduction given in Proposition 3.10.
- $\#(B_k)$ is the number of unnecessary non-trivial obstructions detected by the Non-Commutative Backward Criterion given in Proposition 3.13.
- $\rho = \#(SelObs)/\#(TolObs)$.

Note that $\#(RedGb)$ is an invariant of the ideal which only depends on chosen word ordering. Other numbers in the table rely also on the selection strategy. In our experiments we used the *normal strategy* which first chooses the obstruction whose S-polynomial has the lowest degree and then breaks ties by choosing the obstruction whose S-polynomial has the smallest leading word with respect to the word ordering. The entries in column $\#(TR)$ mean that the Non-Commutative Multiply Criterion and the Non-Commutative Leading Word Criterion have already detected all unnecessary obstructions in the set

$\bigcup_{1 \leq i \leq s'} \text{NTObs}(i, s')$ of newly constructed obstructions. The low ratios ρ in the table indicate that the non-commutative Gebauer-Möller criteria we obtained can detect most unnecessary obstructions during the procedure.

Example 4.2. The following ideals **braid3**, **braid4** and **ufn1h** in the non-commutative polynomial ring $\mathbb{Q}\langle x_1, x_2, x_3, x_4, x_5 \rangle$ are taken from [14], Section 5. More precisely, **braid3** is the ideal generated by the set $\{-x_2x_3x_1 + x_3x_1x_3, x_2x_1x_2 - x_3x_2x_3, x_1x_2x_1 - x_3x_1x_2, x_1^3 + x_1x_2x_3 + x_2^3 + x_3^3\}$, **braid4** is the ideal generated by the set $\{-x_2x_3x_1 + x_3x_1x_3, x_2x_1x_2 - x_3x_2x_3, x_1x_2x_3 - x_3x_1x_2, x_1^3 + x_1x_2x_3 + x_2^3 + x_3^3\}$, and **ufn1h** is the ideal generated by the set $\{-x_4x_5 + x_5x_4, x_4^2 - x_4x_5 - x_3x_5 + x_5x_3, x_3^2 - x_3x_5 - x_2x_5 + x_5x_2, x_2^2 - x_2x_5 - x_1x_5 + x_5x_1, x_1^2 - x_1x_5 - x_3x_4x_5 + x_4x_3x_4, x_3x_4x_3 - x_3x_4x_5, -x_2x_4x_5 + x_4x_2x_4, x_2x_4x_2 - x_2x_4x_5, -x_2x_3x_5 + x_3x_2x_3, x_2x_3x_2 - x_2x_3x_5, x_1x_4x_1 - x_1x_4x_5, -x_1x_3x_5 + x_3x_1x_3, x_1x_3x_1 - x_1x_3x_5, -x_1x_2x_5 + x_4x_1x_4, -x_1x_2x_5 + x_2x_1x_2, x_1x_2x_1 - x_1x_2x_5\}$. These ideals are generated by sets of homogeneous generators. The following table lists the results of the computations of truncated Gröbner bases with respect to LLex on $\langle x_1, x_2, x_3, x_4, x_5 \rangle$ such that $x_1 >_{\text{LLex}} x_2 >_{\text{LLex}} x_3 >_{\text{LLex}} x_4 >_{\text{LLex}} x_5$, via the Improved Buchberger Procedure. The rows **braid3-11**, **braid4-11** and **ufn1h-11** correspond to Gröbner bases truncated at degree 11, while the row **ufn1h-15** corresponds to Gröbner bases truncated at degree 15.

	$\#(Gb)$	$\#(TolObs)$	$\#(SelObs)$	ρ
braid3-11	729	2195	1663	0.7576
braid4-11	417	1675	1150	0.6866
ufn1h-11	360	2638	2406	0.9121
ufn1h-15	892	7138	6558	0.9187

The meaning of the symbols is the same as in Example 4.1. In this experiment we also used the normal strategy. Moreover, since we compute truncated Gröbner bases, we discard those obstructions whose S-polynomial have degrees larger than the truncated degree. In this way we control the total number of non-trivial obstructions. Thus the ratios ρ in the table are higher than the ratios in the table of Example 4.1. However, the non-commutative Gebauer-Möller criteria again detect most unnecessary obstructions during the procedure.

The experimental data in Examples 4.1 and 4.2 show that our generalizations of the Gebauer-Möller criteria, as presented in Propositions 3.7, 3.8 and 3.13, can successfully detect a large number of unnecessary obstructions. We conjecture that our generalizations detect almost all unnecessary obstructions during the Buchberger Procedure.

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